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How to Leverage the Potential of Mathematical Errors

Incorporating a focus on students' mistakes into your instruction can advance their understanding.

By Wendy S. Bray

Telling children that they can learn from their mistakes is common practice. Yet research indicates that many teachers in the United States limit public attention to errors during mathematics lessons (Bray 2011; Santagata 2005). Some believe that drawing attention to errors publicly may embarrass error makers or may be confusing to struggling learners. However, exploratory research suggests that focusing on errors can lead to increased student engagement among struggling learners (Bray 2007). In general, the strategy of publicly analyzing and discussing mathematical errors is thought to promote conceptual understanding (Borasi 1994; Kazemi and Stipek 2001).

This article explores the potential of using errors to advance students' mathematical understanding and presents a framework for infusing a focus on errors into mathematics instruction. The classroom example that follows will anchor the rest of the article. Third graders who were studying initial fraction concepts received this task (see **fig. 1**). Before

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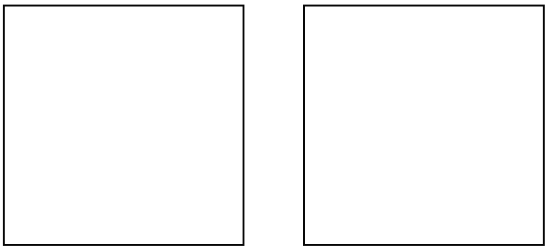
reading further, examine the student solutions presented in **figure 2**, and consider how the various solutions—correct and flawed—might be used instructionally to develop fraction concepts.

FIGURE 1

Third graders received this task during their study of initial fraction concepts.

Three children want to share these two brownies so they each get the same amount.



- Using the brownies above, show one way the children can share the brownies fairly.
- Using fractions, tell how much brownie each child should get. Be ready to justify your answer.
- Draw two more square brownies and show a different way the three children can share fairly. Tell how much brownie each child should get with your new solution.
- Repeat the last step. See how many *different* solutions you can think of.

Using the framework

The balance of this article elaborates how to design and implement lessons to leverage the instructional potential of errors. The framework has been developed through study of other teachers' handling of students' mathematical errors (Bray 2011; Santagata and Bray 2011) and attempts within my own mathematics teaching to incorporate errors. Errors can and do surface in all kinds of mathematics lessons; however, lessons designed to provoke and address students' misconceptions have a greater likelihood of making productive use of errors. Mathematical tasks can be selected for their potential to expose misconceptions, and provision can be made for explicitly attending to errors during instruction.

Selecting mathematical tasks

Problem-based tasks that probe students' depth of understanding of mathematical ideas are useful for bringing students' misconceptions to the surface. Such tasks can usually be approached in different ways and often require students to justify their mathematical thinking. Tasks that are intentionally designed with contexts and numbers that provoke commonly held misconceptions are particularly good at revealing misunderstanding.

In the Three Children Sharing Two Brownies task, the mathematical emphasis is on partitioning brownies into equal parts and identifying the fractions that describe the parts. As children begin to learn about fractions, they tend to think about thirds as three parts rather than three *equal* parts. Consequently, students commonly devise solutions to this task with fractional parts incorrectly labeled, allowing an opportunity to explore this misconception. This task also provokes misconceptions by using numbers that yield a focus on thirds rather than the more easily represented halves or fourths. Finally, the problem context aids in allowing misconceptions to surface by requiring students to draw representations and devise multiple solutions.

Planning for instruction

After selecting mathematical tasks, anticipating the flawed ways that students will approach the tasks is a critical step for leveraging the potential of errors. A teacher can consider in advance how to use particular errors to support the mathematical agenda. With the Sharing Brownies task, the teacher wanted to emphasize how fractional parts are named in relation to their wholes. She anticipated that students would produce solutions with fractional parts named incorrectly and that analysis and discussion of these flawed solutions could promote students' understanding.

As this example suggests, a lesson designed to maximize the instructional potential of errors must make provision for students to analyze and discuss flawed solutions in relation to task specifications and mathematical ideas. A good first step is to have students justify whether a given solution is correct. Further analysis may involve considering the logic behind an error and determining how to revise and correct it.

Flawed solutions that are selected for public

discussion can come from the class or from the teacher. Students' initial analysis of an error can occur independently, between partners, in small groups, or through whole-class discussion. However, two essentials elements are that (1) students actively engage in attempts to unpack the mathematics underlying the errors and (2) the teacher facilitates public discussion of ideas such that key concepts are emphasized. During the planning stage, therefore, identify how examining specific errors will illuminate the mathematical agenda and make provision for students to grapple with errors as a community of learners. Last, to inform public discussion, have a plan for determining the flawed (and correct) ways that students are approaching tasks (Smith and Stein 2011).

Implementing the lesson

After planning a lesson with a focus on errors, the challenge for teachers is to make on-the-spot adjustments to the lesson in response to the actual understandings and misunderstandings he or she observes. To use errors effectively, teachers can take specific actions as students work on tasks and publicly discuss solutions.

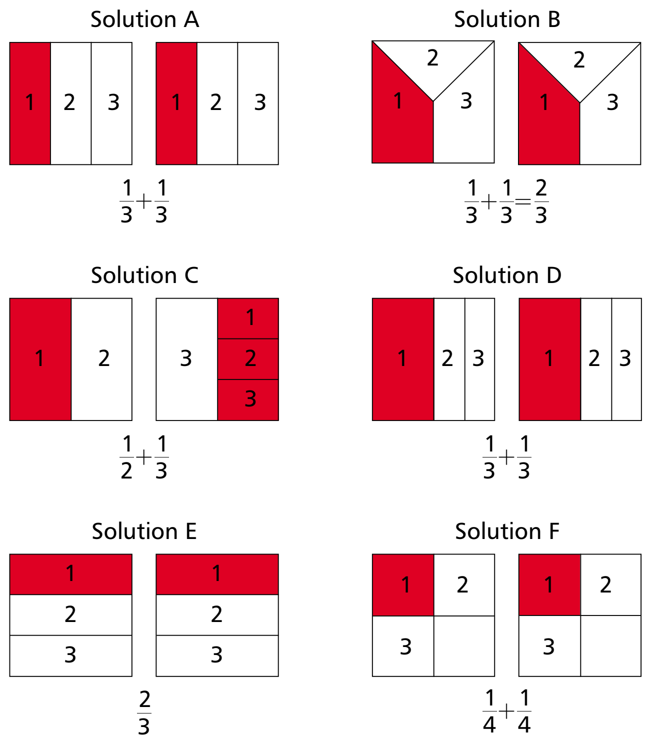
While the third graders began working on the Sharing Brownies task, the teacher circulated around the room to ensure that all students understood and were attempting the task. Simultaneously, she sought to gauge how students' solutions compared with what she had anticipated. After most students had two solutions, the teacher directed table groups to share their work with attention to the following questions:

- What different solutions did your table generate?
- Does each solution show a fair way for three children to share two brownies?
- Did you use fractions correctly to name parts of the brownies?

As the teacher visited table groups, she listened to students' reasoning with particular attention to errors and underlying misconceptions. She also decided on the solutions—correct and flawed—to include in a public discussion and settled on a plan for how to use these solutions to emphasize part-whole fraction relationships. Then she requested that particular solutions be re-created on small white boards for sharing in a public discussion.

FIGURE 2

The Sharing Two Brownies task uses numbers that focus on thirds rather than halves or fourths. Requiring students to draw representations, devise multiple solutions, and label fractional parts, the task often exposes misconceptions.



By the time the teacher had distributed white boards, table groups often had identified either their errors or their uncertainty about the correctness of particular solutions. The teacher praised students for these observations and, if she wanted an error to be shared publicly, suggested that discussion of the solution could help the class better understand fractions. She also sought the permission of error makers to share their solutions publicly.

Discussing the solutions publicly

The public discussion of the Sharing Brownies task was organized in two parts. First, the teacher had the class examine solution C (see **fig. 2**) because she had noticed that many students initially split the two brownies in half and then struggled to make fair shares. Solution C

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successfully makes fair shares by partitioning the two brownies into halves and then partitioning one-half of a brownie into three equal parts. But then the solution incorrectly identifies the small parts as thirds rather than sixths. The teacher viewed this error as an opportunity to have students think deeply about how fractional parts are named. Specifically, she wanted students to see that they could determine the fractional amount of a part by filling the whole with parts of the same size and shape (see **fig. 3**).

The teacher first asked Amy—the error maker—to explain her solution C to the class. Before Amy began, the teacher directed the rest of the students to listen to Amy's explanation to *understand* her approach and *evaluate* her solution. Amy recounted her strategy and explained that she split the last half into thirds.

The teacher asked the class to show thumbs up or down on whether Amy's strategy led to fair shares. Observing a roomful of quick thumbs up, she continued, "In this solution, are fractions used correctly to name parts of brownies?" Students were more hesitant with this question and displayed thumbs up, down, and sideways (i.e., unsure). After several students shared their reasoning, the teacher guided the class to agree that the one-half piece of the second brownie was in thirds but that *one-third* did not describe the fractional part of a whole brownie represented by each small piece. Next, the teacher challenged students to work with their shoulder partners to figure out the fractional part of a brownie represented by each small piece. To encourage strategic thinking, she reminded students that they knew the one-half-size piece was one-half because they could fit two of them into a whole brownie, and she wondered aloud how many of the small pieces could fit into a whole brownie. After a few minutes, this part of the discussion concluded with two partnerships explaining how they determined the small piece was one-sixth of a brownie by using filling strategies. As students shared, the teacher directly engaged those who still seemed confused by asking them to restate why the small part was one-sixth of a brownie.

To open the second part of the public discussion, the teacher displayed the white boards detailing solutions A, B, D, and E and asked the class to identify similarities. Students noted that these solutions all involved cutting brownies in three parts and that they all showed either one-third plus one-third, or two-thirds (which the class recognized as equivalent). The teacher then directed students to talk with their shoulder partner about the following questions:

- Which solutions show brownies split into thirds? How do you know?
- For the solutions that do not show thirds, what fractions are they showing?

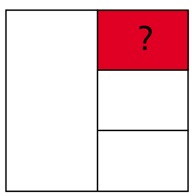
When public discussion reconvened, students quickly identified with appropriate justification that solutions A and E showed brownies split into thirds and that solution D did not. Students were also able to use the filling strategy to justify that solution D brownies were partitioned into one-half, one-fourth, and one-fourth.

At this point, most students were still unsure of the correctness of solution B. One partnership, however, used a picture to justify that the brownie was not in thirds (see **fig. 4**). Victoria said, "This [triangle-shaped] piece is one-fourth because you can make it four times in the square. These two [trapezoid-shaped parts] are bigger than one-fourth." Victoria and her partner had not yet determined the fractional name for the trapezoid-shaped parts, but they

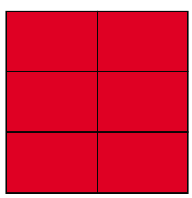
FIGURE 3

The teacher wanted students to understand how to determine the fractional amount of a part by filling the whole with parts of the same size and shape.

What fraction of the brownie is this part?



To solve, "fill" the whole brownie with parts of the same shape and size.



The part is 1/6 of the whole brownie because it took six congruent parts to "fill" the brownie.

had determined that the trapezoids could not be iterated to fill the whole. After having multiple students summarize their understanding of Victoria's explanation, the teacher challenged the class to figure out how they might add additional lines to Victoria's picture to help discover the fraction of a brownie represented by the trapezoid-shaped part. Students again worked with shoulder partners and then discussed their strategies as a class.

Focusing on the math concepts

During the public discussion phase, two tenets can be used to guide working with errors. First, it is important to keep the focus of discussion on the conceptual underpinnings and key mathematical lesson of a given error. To accomplish this, the teacher must be clear in her own mind about the mathematical concepts or strategies to emphasize and the trajectory along which those understandings are expected to develop. With the Sharing Brownies task, the focus was on developing students' capacity to determine and justify fractions that named specified parts of a whole. In the public discussion, students were first led to think about flawed solutions C and D, for which the fractional parts could be justified using the filling strategy. Then they were challenged to consider the more difficult flawed solution B, for which the fractions could not be determined without partitioning the figure further.

To keep the emphasis on mathematical concepts, students were challenged to justify why certain aspects of the solutions were correct or incorrect. When it makes sense, students can also be guided to revise and correct flawed solutions in ways that retain aspects of the original strategy. Finally, the teacher can summarize students' ideas to ensure that they connect explicitly to her mathematical agenda for the discussion.

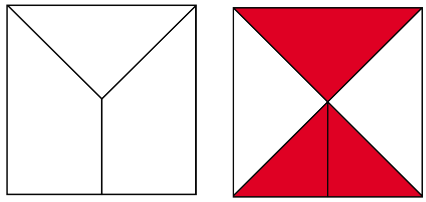
Fostering a community of learners

The second principle is a commitment to having students interact with errors during public discussion as a *community of learners*. In a community of learners, students actively engage in working collaboratively to grapple with mathematical ideas. To foster such an approach during discussions of flawed solutions, teachers must avoid immediately identifying errors and instead orchestrate discussion such that students uncover and unpack errors. As students engage with errors, maintaining a classroom culture that values learning from mistakes and honors error makers is imperative.

Sustaining active student engagement is also essential. One strategy is to explicitly assign students a job to guide their examination of errors. Students might be directed to compare, to evaluate, to develop questions, or to explain. In the Sharing Brownies lesson, the teacher first assigned students the job of listening to Amy's flawed solution to understand and evaluate

FIGURE 4

Although Victoria and her partner did not know the fractional name for the trapezoids, they knew that the trapezoids could not be iterated to fill the whole.



Original figure

Drawing to justify that the triangle-shaped piece is 1/4 rather than 1/3

Maximizing productive use of errors

Lessons focused on errors can involve students in working on mathematical tasks independently, with a partner, or in small groups. Regardless of the organization of student work time, a crucial teacher action for maximizing the productive use of errors is to find out how students are approaching tasks and the errors they are making. The teacher can then use this information to formulate a plan for public discussion. Specifically, three actions are important: Identify the—

1. flawed and correct solutions that will be focused on in the public discussion;
2. order and format in which these solutions will be shared; and
3. mathematical points to be made with each.

Additionally, for students to be comfortable sharing their (sometimes flawed) work, promoting a classroom culture that is supportive of intellectual risk-taking is essential.

TABLE 1

Use these four strategies to leverage the potential of your students' mathematical errors.

Phase	Strategies
Selecting mathematical tasks	→ Identify the mathematical focus and related misconceptions. → Use problem-based tasks that emphasize the mathematical focus. → Tweak task features (contexts, numbers) to provoke misconceptions.
Planning for instruction	→ Anticipate students' errors and identify underlying misconceptions. → Consider how errors can be used instructionally to illuminate mathematical ideas. → Plan a way to find out how students are approaching tasks. → Make provision for students to analyze and publicly discuss flawed solutions.
As students work on tasks	→ Promote a culture of intellectual risk taking. → Find out how students are approaching tasks and the errors they are making. → Decide on a plan for public discussion (solutions to include, order, purpose).
During public discussion	→ Engage students as a community of learners to analyze flawed solutions. → Emphasize the conceptual lessons of errors. → Honor error makers and learning from mistakes.

its accuracy. Later she gave students the job of working with a shoulder partner to determine the fractions represented in Amy's solution. The expectation of active engagement and the explicit direction on how to engage both seem highly related to the degree to which students attempt the hard work of analyzing and learning from errors.

Getting started with errors

When planning for and teaching lessons with a focus on errors, concentrate on four phases of the process: First, select mathematical tasks for their potential to expose students' misconceptions. Then plan lessons with a consideration of how errors might be used to advance students' mathematical understanding. As students work, be attentive to their solutions and make a plan for incorporating errors into public discussion. Finally, engage students as a community of learners to analyze and revise errors such that key mathematical concepts are illuminated. (See the summary of strategies in **table 1**.)

Undergirding the successful use of students' errors to promote learning is a classroom environment in which children feel safe sharing their (sometimes flawed) mathematical ideas and solutions. For children to undertake this kind of intellectual risk taking, it is essential to establish and maintain a climate of respect and supportiveness in the classroom. (See Chapin, O'Connor, and Anderson 2009 for a detailed description of how to establish and maintain these and other classroom norms that support productive mathematical discourse.) To specifically prepare students for learning through examination of errors, talk with students about how mistakes are a natural and important part of the learning process. For instance, the only way to learn how to hit a baseball is by taking swing after swing and making adjustments based on what is learned from the hits and misses. In a school context, we learn to read by persisting through our difficulties decoding text and learning from them.

In addition to helping students view the examination of errors in a positive light, have explicit conversations about how to engage respectfully in discussing a classmate's flawed work. One possibility for initially working on this skill is for the teacher to present a flawed solution and say, "This is one way that I have seen children solve this problem." As students publicly examine the flawed solution, prompt them to reflect on appropriate ways to convey ideas.

When asking children to share their flawed solutions, take additional steps to respect and honor error makers. Error makers are less likely to feel put on the spot if they are aware that their solutions may contain errors and if they have some control over the process of their work being shared. Students can gain awareness of errors through informal comparison of solutions with peers. Teachers can give error makers control by asking permission to include their work in public discussion and by giving them a choice about whether to personally explain their mathematical approach. Some students are reluctant to talk publicly about their work if they suspect it is flawed but are happy to have the class analyze and revise their flawed

solutions. Teachers can further honor error makers by recognizing the logic inherent in their flawed mathematical thinking and by emphasizing that their mistakes provide an opportunity for the class to better understand mathematics.

Benefiting from the framework

As teachers begin to think about mathematics instruction with a focus on misconceptions, they become better able to anticipate students' errors and respond to them in mathematically productive ways. For students, placing an emphasis on learning from mistakes correlates to participation in mathematical discourse that is more evenly distributed, as well as to greater understanding of mathematics. One teacher commented,

> When I include [an analysis of] wrong answers, there is a place for everyone in the discussion, especially the students who don't quite get it.

My hope is that this framework can help more teachers think about how to make learning from mistakes a more explicit and valued part of mathematics instruction.

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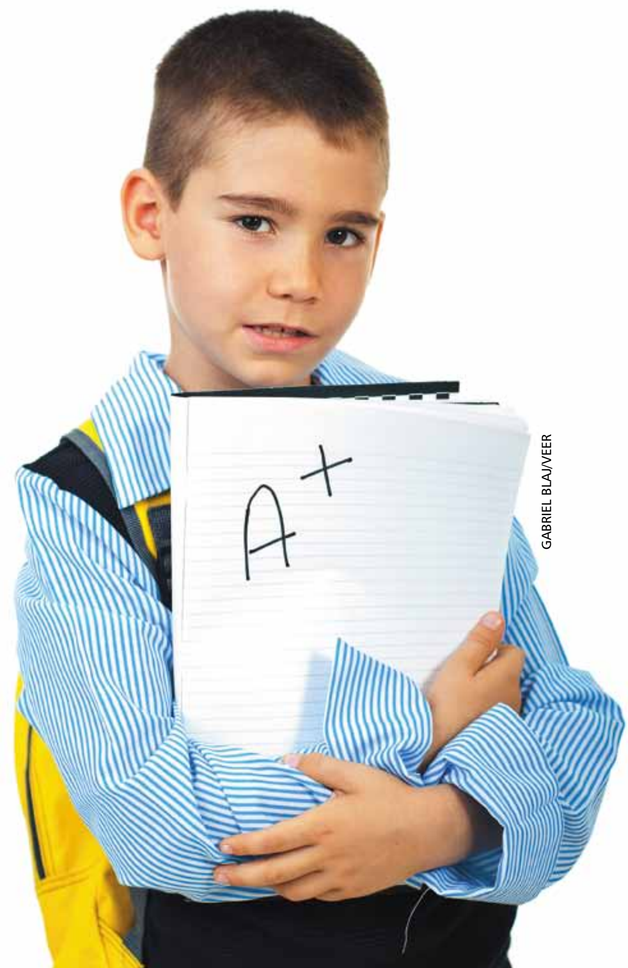

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